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CONVEXITY OF MINIMAL DOMINATING AND TOTAL DOMINATING FUNCTIONS OF CORONA PRODUCT GRAPH OF A PATH WITH A STAR

[1]Makke Siva Parvathi [2]Bommireddy Maheswari

Abstract: 'Domination in graphs' has been studied extensively and at present it is an emerging area of research in graph theory. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [1,2]. Product of graphs occurs naturally in discrete mathematics as tools in combinatorial constructions. They give rise to an important classes of graphs and deep structural problems. In this paper we study the dominating and total dominating functions of corona product graph of a cycle with a complete graph.

Index Terms: Corona Product, Path, Star, Dominating function, Total dominating function.

1. INTRODUCTION

Domination Theory is an important branch of Graph Theory that has many applications in Engineering, Communication Networks, mobile computing, resource allocation, telecommunication and many others. Allan, R.B. and Laskar, R.[3], Cockayne, E.J. and Hedetniemi, S.T. [4] have studied various domination parameters of graphs.

Recently, dominating functions in domination theory have received much attention. The concepts of total dominating functions and minimal total dominating functions are introduced by Cockayne et al. [5]. Jeelani Begum, S. [6] has studied some total dominating functions of Quadratic Residue Cayley graphs.

Frucht and Harary [7] introduced a new product on two graphs G_1 and G_2, called corona product denoted by $G_1 \odot G_2$. The object is to construct a new and simple operation on two graphs G_1 and G_2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G_1 and of G_2 .

The authors have studied some dominating functions of corona product graph of a path with a star [8]. In this paper we discuss the convexity of minimal dominating and total dominating functions of corona product graph of a path with a star.

2. CORONA PRODUCT OF P_n AND $K_{1,m}$

The **corona product** of a path P_n with star $K_{1,m}$ is a graph obtained by taking one copy of a n – vertex path P_n and n copies of $K_{1,m}$ and then joining the i^{th} vertex of P_n to every vertex of i^{th} copy of $K_{1,m}$ and it is denoted by $P_n \odot K_{1,m}$.

We require the following theorem whose proof can be found in Siva Parvathi, M. [8].

Theorem 2.1: The degree of a vertex v_i in $G = P_n \odot K_{1,m}$ is given by

$$d(v_i) = \begin{cases} m+3, & if \quad v_i \in P_n \ and \quad 2 \le i \le (n-1), \\ m+2, & if \quad v_i \in P_n \ and \quad i = 1 \ or \ n, \\ m+1, & if \ v_i \in K_{1,m} \ and \ v_i \ is \ in \ first \ partition, \\ 2, & if \ v_i \in K_{1,m} \ and \ v_i \ is \ in \ second \ partition. \end{cases}$$

3. CONVEXITY OF MINIMAL DOMINATING FUNCTIONS

A study of convexity and minimality of dominating functions (MDFs) are given in Cockayne et al.[9] and Yu[10]. Rejikumar [11] developed a necessary and sufficient condition for the convex combination of two MDFs to be again a MDF. Jeelani Begum [6] studied convexity of MDFs of Quadratic Residue Cayley Graphs.

In this section we discuss the convexity of minimal dominating functions of corona product graph $G = P_n \odot K_{1,m}$. First we define the convex combination of functions and prove some results on the convexity of MDFs of G.

Definition: Let $G\ (V, E)$ be a graph. Let f and g be two functions from V to $[0, 1]$ and $\lambda \in (0, 1)$. Then the function $h : V \rightarrow [0,1]$ defined by $h(v) = \lambda f(v) + (1-\lambda)\ g(v)$ is called a **convex combination** of f and g.

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[1]Dept. of Applied Mathematics, SPMVV, Tirupati, A.P.
Email id:parvathimani2008@gmail.com
[2]Dept. of Applied Mathematics, SPMVV, Tirupati, A.P.
Email id:maherahul.55@gmail.com

Definition: Let $G(V,E)$ be a graph. A subset D of V is said to be a **dominating set** (DS) of G if every vertex in $V-D$ is adjacent to some vertex in D.

A dominating set D is called a **minimal dominating set** (MDS) if no proper subset of D is a dominating set of G.

Definition: The **domination number** of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma(G)$.

Definition: Let $G(V,E)$ be a graph. A function $f: V \to [0,1]$ is called a **dominating function** (DF) of G if $f(N[v]) = \sum_{u\in N[v]} f(u) \geq 1, \ for\ each\ v \in V.$

Definition: Let f and g be functions from V to $[0,1]$. We define $f < g$ if $f(u) \leq g(u)$ for all $u \in V$, with strict inequality for at least one vertex $u \in V$.

A dominating function f of G is called a **minimal dominating function** (MDF) if for all $g < f$, g is not a dominating function.

Theorem 3.1: Let D_1 and D_2 be two MDSs of $G = P_n \odot K_{1,m}$. Let $f_1: V \to [0,1]$ and $f_2: V \to [0,1]$ be defined by

$$f_1(v) = \begin{cases} 1, & \text{if } v \in D_1, \\ 0, & \text{otherwise.} \end{cases}$$

and $$f_2(v) = \begin{cases} 1, & \text{if } v \in D_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then the convex combination of f_1 and f_2 becomes a MDF of $G = P_n \odot K_{1,m}$.

Proof: Let D_1 and D_2 be two MDSs of G.
Let f_1, f_2 be two functions defined as in the hypothesis.
These functions are MDFs of $G = P_n \odot K_{1,m}$[8].

Let $h(v) = \alpha f_1(v) + \beta f_2(v), \ where \ \ \alpha + \beta = 1 \ and \ \ 0 < \alpha < 1, \ 0 < \beta < 1.$

Case 1: Suppose $D_1 \cap D_2 \neq \varphi$.
For $v \in V$, the possible values of $h(v)$ are

$$h(v) = \begin{cases} \alpha, & \text{if } v \in D_1 - D_2, \\ \beta, & \text{if } v \in D_2 - D_1, \\ \alpha+\beta, & \text{if } v \in D_1 \cap D_2, \\ 0, & \text{otherwise} . \end{cases}$$

Then

$$\sum_{u\in N[v]} h(u) = s\alpha + t\beta, \text{ if s - vertices of } D_1 \text{ and t - vertices of } D_2 \text{ are in } N[v].$$

So $$\sum_{u\in N[v]} h(u) \geq 1, \ \forall \ \ v \in V.$$

This implies that h is a DF.

Now we check for the minimality of h.

Define $g: V \to [0,1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_i \in D_1 \cap D_2, \\ \alpha+\beta, & \text{if } v \in (D_1 \cap D_2) - \{v_i\}, \\ \alpha, & \text{if } v \in D_1 - D_2, \\ \beta, & \text{if } v \in D_2 - D_1, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < r < 1$.

Since strict inequality holds at the vertex $v_i \in V$, it follows that $g < h$.

Then $\sum_{u\in N[v]} g(u)$

$$= \begin{cases} r, & \text{if } v \in i^{th} \text{ copy of } K_{1,m} \text{ in G,} \\ s\alpha + t\beta + r, & \text{if s-vertices of } D_1, \text{ t-vertices of } D_2 \text{ and } v_i \text{ are in } N[v], \\ s\alpha + t\beta, & \text{if s-vertices of } D_1 \text{ and t-vertices of } D_2 \text{ are in } N[v]. \end{cases}$$

This implies that

$\sum_{u \in N[v]} g(u) = r < 1$, for the vertices in the i^{th} copy of $K_{1,m}$ in G.

So g is not a DF.

Since g is taken arbitrarily, it follows that there exists no $g < h$ such that g is a DF.

Thus h is a MDF.

Case 2: Suppose $D_1 \cap D_2 = \varphi$.

For $v \in V$, the possible values of $h(v)$ are

$$h(v) = \begin{cases} \alpha, & \text{if } v \in D_1, \\ \beta, & \text{if } v \in D_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$\sum_{u \in N[v]} h(u) = s\alpha + t\beta$, if s - vertices of D_1 and t - vertices of D_2 are in $N[v]$.

So $\sum_{u \in N[v]} h(u) \geq 1, \ \forall \ v \in V.$

This implies that h is a DF.

Now we check for the minimality of h.

Define $g : V \to [0,1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_i \in D_1, \\ \alpha, & \text{if } v \in D_1 - \{v_i\}, \\ \beta, & \text{if } v \in D_2, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < r < \alpha$.

Since strict inequality holds at the vertex $v_i \in V$, it follows that $g < h$.

Then

$$g(v) = \begin{cases} r + \beta, & \text{if } v \in i^{th} \text{ copy of } K_{1,m} \text{ in } G, \\ s\alpha + t\beta + r, & \text{if s - vertices of } D_1, \text{ t - vertices of } D_2 \text{ and } v_i \text{ are in } N[v], \\ s\alpha + t\beta, & \text{if s - vertices of } D_1 \text{ and t - vertices of } D_2 \text{ are in } N[v]. \end{cases}$$

This implies that $\sum_{u \in N[v]} g(u) = r + \beta < \alpha + \beta = 1$, for the vertices in the i^{th} copy of $K_{1,m}$ in G.

So g is not a DF.

Since g is taken arbitrarily, it follows that there exists no $g < h$ such that g is a DF.

Thus h is a MDF. ■

4. CONVEXITY OF MINIMAL TOTAL DOMINATING FUNCTIONS

The concepts of total dominating functions (TDFs) and minimal total dominating functions (MTDFs) are introduced by Cockayne et al. [12]. A study of convexity and minimality of TDFs are given in Cockayne et al. [9, 12]. Cockayne et al.[9] obtained a necessary and sufficient condition for the convex combination of two minimal total dominating functions to be again a minimal total dominating function.

The authors have studied minimal total dominating functions of $G = P_n \odot K_{1,m}$ [8]. In this section we consider minimal total dominating functions of corona product graph $G = P_n \odot K_{1,m}$ and discuss their convexity .

Definition: Let $G(V, E)$ be a graph without isolated vertices. A subset T of V is called a **total dominating set** (TDS) if every vertex in V is adjacent to at least one vertex in T.

If no proper subset of T is a total dominating set, then T is called a **minimal total dominating set** (MTDS) of G.

Definition: The minimum cardinality of a MTDS of G is called a **total domination number** of G and is denoted by $\gamma_t(G)$.

Definition: Let $G(V, E)$ be a graph. A function $f : V \to [0,1]$ is called a **total dominating function** (TDF) of G if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1,$ *for each* $v \in V$.

Definition: Let f and g be functions from V to $[0,1]$. We define $f < g$ if $f(u) \leq g(u)$ for all $u \in V$, with strict inequality for at least one vertex $u \in V$.

A TDF f of G is called a **minimal total dominating function** (MTDF) if for all $g < f$, g is not a TDF.

Theorem 4.1: Let T_1 and T_2 be two MTDSs of $G = P_n \odot K_{1,m}$. Let $f_1 : V \to [0,1]$ and $f_2 : V \to [0,1]$ be defined by

$$f_1(v) = \begin{cases} 1, & \text{if } v \in T_1, \\ 0, & \text{otherwise.} \end{cases}$$

and $$f_2(v) = \begin{cases} 1, & \text{if } v \in T_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then the convex combination of f_1 and f_2 becomes a MTDF of $G = P_n \odot K_{1,m}$.

Proof: Consider the graph $G = P_n \odot K_{1,m}$ with vertex set V.

Let T_1 and T_2 be two MTDSs of G. Let f_1, f_2 be two functions defined as in the hypothesis.

By Theorem 3.3.2, these functions are MTDFs of $G = P_n \odot K_{1,m}$.

Let $h(v) = \alpha f_1(v) + \beta f_2(v),\ where\ \ \alpha + \beta = 1$ $and\ \ 0 < \alpha < 1,\ 0 < \beta < 1.$

Case 1: Suppose $T_1 \cap T_2 \neq \varphi$.

For $v \in V$, the possible values of $h(v)$ are

$$h(v) = \begin{cases} \alpha, & \text{if } v \in T_1 - T_2, \\ \beta, & \text{if } v \in T_2 - T_1, \\ \alpha + \beta, & \text{if } v \in T_1 \cap T_2, \\ 0, & \text{otherwise .} \end{cases}$$

Then

$$\sum_{u \in N(v)} h(u) = s\alpha + t\beta, \quad \text{if s - vertices of } T_1 \text{ and t - vertices of } T_2 \text{ are in N(v).}$$

Therefore $\sum_{u \in N(v)} h(u) \geq 1,\ \forall\ \ v \in V.$

This implies that h is a TDF.

Now we check for the minimality of h.

Define $g : V \to [0,1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_i \in T_1 \cap T_2, \\ \alpha + \beta, & \text{if } v \in (T_1 \cap T_2) - \{v_i\}, \\ \alpha, & \text{if } v \in T_1 - T_2, \\ \beta, & \text{if } v \in T_2 - T_1, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < r < 1$.

Since strict inequality holds at the vertex $v_i \in T_1 \cap T_2$, it follows that $g < h$.

Then $\sum_{u \in N(v)} g(u)$

$$= \begin{cases} r, & \text{if } v \in i^{th} \text{ copy of } K_{1,m} \text{ in G,} \\ s\alpha + t\beta + r, & \text{if s - vertices of } T_1\text{, t - vertices of } T_2 \text{ and } v_i \text{ are in N(v),} \\ s\alpha + t\beta, & \text{if s - vertices of } T_1 \text{ and t - vertices of } T_2 \text{ are in N(v).} \end{cases}$$

This implies that $\sum_{u \in N(v)} g(u) = r < 1$, for the vertices in the i^{th} copy of $K_{1,m}$ $in\ G.$

So g is not a TDF.

Since g is taken arbitrarily, it follows that there exists no $g < h$ such that g is a TDF.

Thus h is a MTDF.

Case 2: Suppose $T_1 \cap T_2 = \varphi$.

For $v \in V$, the possible values of $h(v)$ are

$$h(v) = \begin{cases} \alpha, & \text{if } v \in T_1, \\ \beta, & \text{if } v \in T_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{u \in N(v)} h(u) = s\alpha + t\beta, \quad \text{if s - vertices of } T_1 \text{ and t - vertices of } T_2 \text{ are in N(v).}$$

Therefore $\sum_{u \in N(v)} h(u) \geq 1,\ \forall\ \ v \in V.$

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This implies that h is a TDF.

Now we check for the minimality of h.

Define $g : V \to [0,1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_i \in T_1, \\ \alpha, & \text{if } v \in T_1 - \{v_i\}, \\ \beta, & \text{if } v \in T_2, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < r < \alpha$.

Since strict inequality holds at the vertex $v_i \in T_1$, it follows that $g < h$.

Then

$$\sum_{u \in N(v)} g(u) = \begin{cases} r+\beta, & \text{if } v \in i^{th} \text{ copy of } K_{1,m} \text{ in } G, \\ s\alpha + t\beta + r, & \text{if } s\text{-vertices of } T_1, t\text{-vertices of } T_2 \text{ and } v_i \text{ are in } N(v), \\ s\alpha + t\beta, & \text{if } s\text{-vertices of } T_1 \text{ and } t\text{-vertices of } T_2 \text{ are in } N(v). \end{cases}$$

This implies that $\sum_{u \in N(v)} g(u) = r + \beta < \alpha + \beta = 1$, for the vertices in the i^{th} copy of $K_{1,m}$ in G.

So g is not a TDF.

Since g is taken arbitrarily, it follows that there exists no $g < h$ such that g is a TDF.

Thus h is a MTDF. ■

5. CONCLUSION

It is interesting to study the convexity of minimal dominating and total dominating functions of corona product graph of a cycle with a complete graph. This work gives the scope for an extensive study of dominating functions in general of these graphs.

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